THE FUNDAMENTAL EQUATIONS OF THE GENERAL THEORY OF THIN ELASTIC SHELLS IN TERMS OF STRESSES

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The progress of the theory of shells takes place in the direction of development of approximate computational procedures adapted to the treatment of certain types of problems encountered in various technical applications. Particular attention is being paid in this development also to the question of representing the methods of analysis by means of sufficiently simple and clearly understandable steps, leaving to the designer the possibility of unobstructed orientation in making an appropriate choice among the available approximate ways of computation. With this circumstance in mind, one notices easily that the equations of the general theory of shells, taken in their usual form and containing the non-associated quantities T_1, T_2, S, M_1, M_2, H, are not convenient for derivation of approximate theories, since the simplification of the equations mentioned, accomplished by omission of certain terms considered negligible in comparison with some other terms, is very often a speculative operation, understandable only to a narrow circle of specialists. The process of deriving approximate theories is considerably improved if instead of the stress resultants and couples we use more unified quantities, such as stresses acting in the extreme layers of the shell thickness, as fundamental unknown quantities of the problem. In the following we offer a particular version of presenting the general equations of the theory of shells in terms of the stresses just mentioned. It is shown that in this representation both the derivation of the fundamental equations and the statement of a number of known theoretical results are clearer and simpler. In the process of setting up the equations we use the generally accepted notations for the quantities appearing in the analysis and adopted also in [1] to [5]. Small quantities will be treated in all equations with the accuracy to $1 + \lambda + \lambda^2 = 1 + \lambda$, where λ is the ratio of half-thickness of the shell to the radius of curvature of the latter.

1. Equations of statics. Assuming the axes of the coordinates α, β, z to be directed along the lines of the principal curvatures and the outer normal to the middle surface, respectively, and using the law of linear distribution of the fundamental stresses across the shell thickness, we have

$$\sigma_\alpha = \sigma_1 + m_1 \frac{z}{h}, \qquad \sigma_\beta = \sigma_2 + m_2 \frac{z}{h}, \qquad \sigma_{\alpha\beta} = \sigma_{12} + m_{12} \frac{z}{h} \tag{1.1}$$

where σ_1, σ_2 and σ_{12} are the normal stresses and the shear stress, respectively, at a point of the middle surface, while m_1, m_2 and m_{12} are the normal stresses and the shear stress, respectively, at the extreme points $z = \pm h$, produced by bending and twist, respectively. The total stresses at these extreme points will be

$$\begin{aligned} &\sigma_\alpha^+ = \sigma_1 + m_1, \qquad \sigma_\beta^+ = \sigma_2 + m_2, \qquad \sigma_{\alpha\beta}^+ = \sigma_{12} + m_{12} \\ &\sigma_\alpha^- = \sigma_1 - m_1, \qquad \sigma_\beta^- = \sigma_2 - m_2, \qquad \sigma_{\alpha\beta}^- + \sigma_{12} - m_{12} \end{aligned} \tag{1.2}$$

Substituting (1.1) into the expressions for the stress resultants and couples, we find

$$\begin{aligned} T_1 &= \int \sigma_\alpha (1 + k_2 z)\, dz, \qquad & M_1 &= \int z\sigma_\alpha (1 + k_2 z)\, dz \\ T_2 &= \int \sigma_\beta (1 + k_1 z)\, dz, \qquad & M_2 &= \int z\sigma_\beta (1 + k_1 z)\, dz \\ S_1 &= \int \sigma_{\alpha\beta} (1 + k_2 z)\, dz, \qquad & H_1 &= \int z\sigma_{\alpha\beta} (1 + k_2 z)\, dz \\ S_2 &= \int \sigma_{\alpha\beta} (1 + k_1 z)\, dz, \qquad & H_2 &= \int z\sigma_{\alpha\beta} (1 + k_1 z)\, dz \end{aligned} \tag{1.3}$$

$$\begin{aligned} &T_1 = 2h[\sigma_1 + \tfrac{1}{3} m_1\lambda_2], \qquad M_1 = \tfrac{2}{3} h^2 [m_1 + \sigma_1\lambda_2] \qquad \left(\lambda_1 = \frac{h}{R_1} = hk_1\right) \\ &T_2 = 2h[\sigma_2 + \tfrac{1}{3} m_2\lambda_1], \qquad M_2 = \tfrac{2}{3} h^2 [m_2 + \sigma_2\lambda_1] \qquad \left(\lambda_2 = \frac{h}{R_2} = hk_2\right) \\ &S_1 = 2h[\sigma_{12} + \tfrac{1}{3} m_{12}\lambda_2], \qquad H_1 = \tfrac{2}{3} h^2 [m_{12} + \sigma_{12}\lambda_2] \\ &S_2 = 2h[\sigma_{12} + \tfrac{1}{3} m_{12}\lambda_1], \qquad H_2 = \tfrac{2}{3} h^2 [m_{12} + \sigma_{12}\lambda_1] \end{aligned} \tag{1.4}$$

Substitution of (1.4) into the well-known equilibrium equations for a shell element gives

$$\begin{aligned} &\frac{1}{A}\frac{\partial T_1}{\partial\alpha} + \frac{1}{B}\frac{\partial S_2}{\partial\beta} + \frac{T_1 - T_2}{AB}\frac{\partial B}{\partial\alpha} + \frac{S_1 + S_2}{AB}\frac{\partial A}{\partial\beta} + k_1 N_1 + q_1 = 0 \\ &\frac{1}{A}\frac{\partial S_1}{\partial\alpha} + \frac{1}{B}\frac{\partial T_2}{\partial\beta} + \frac{T_2 - T_1}{AB}\frac{\partial A}{\partial\beta} + \frac{S_1 + S_2}{AB}\frac{\partial B}{\partial\alpha} + k_2 N_2 + q_2 = 0 \\ &\frac{1}{A}\frac{\partial N_1}{\partial\alpha} + \frac{1}{B}\frac{\partial N_2}{\partial\beta} + \frac{N_1}{AB}\frac{\partial B}{\partial\alpha} + \frac{N_2}{AB}\frac{\partial A}{\partial\beta} - k_1 T_1 - k_2 T_2 + q_3 = 0 \end{aligned} \tag{1.5}$$

$$\frac{1}{A}\frac{\partial M_1}{\partial\alpha}+\frac{1}{B}\frac{\partial H_2}{\partial\beta}+\frac{M_1-M_2}{AB}\frac{\partial B}{\partial\alpha}+\frac{H_1+H_2}{AB}\frac{\partial A}{\partial\beta}=N_1$$
$$\frac{1}{A}\frac{\partial H_1}{\partial\alpha}+\frac{1}{B}\frac{\partial M_2}{\partial\beta}+\frac{M_2-M_1}{AB}\frac{\partial A}{\partial\beta}+\frac{H_1+H_2}{AB}\frac{\partial B}{\partial\alpha}=N_2 \qquad (1.6)$$
$$S_1-S_2+k_1H_1-k_2H_2=0$$

It is easily seen that the last (sixth) equation is fulfilled identically. Turning to the other equations, we find that after elimination of the transverse forces only the stresses σ_i, m_i will appear in them. In the interest of brevity we introduce the following symbols:

$$L_1(x,\ y)=\frac{1}{A}\frac{\partial x}{\partial\alpha}+\frac{y}{AB}\frac{\partial A}{\partial\beta},\qquad L_2(x,\ y)=\frac{1}{B}\frac{\partial x}{\partial\beta}+\frac{y}{AB}\frac{\partial B}{\partial\alpha} \qquad (1.7)$$

For $x = L_1(\Phi,\ 0)$, $y = L_2(\Phi,\ 0)$ we have

$$L_1(x,\ y)+L_2(y,\ x)=\nabla^2\Phi,\qquad L_1(y,-x)=L_2(x,-y) \qquad (1.8)$$

where ∇^2 represents the Laplace operator in curvilinear coordinates. Equations (1.5) assume the form

$$L_1(\sigma_1,\ 2\sigma_{12})+L_2(\sigma_{12},\ \sigma_1-\sigma_2)+\frac{\lambda_1+\lambda_2}{3}L_1(m_1, 2m_{12})+$$
$$+\frac{2\lambda_1}{3}L_2(m_{12},\ m_1-m_2)+\frac{q_1}{2h}=0$$
$$L_2(\sigma_2,\ 2\sigma_{12})+L_1(\sigma_{12},\ \sigma_2-\sigma_1)+\frac{\lambda_1+\lambda_2}{3}L_2(m_2,\ 2m_{12})+ \qquad (1.9)$$
$$+\frac{2\lambda_2}{3}L_1(m_{12},\ m_1-m_2)+\frac{q_2}{2h}=0$$
$$\frac{h}{3}\left[L_1(n_1,\ n_2)+L_2(n_2,\ n_1)\right]-\frac{\sigma_1}{R_1}-\frac{\sigma_2}{R_2}+\frac{q_3}{2h}=0\qquad\left(n_1=\frac{3}{2h^2}N_1,\quad n_2=\frac{3}{2h^2}N_2\right)$$

For the quantities n_1, n_2 we obtain from (1.6) the expressions

$$n_1=L_1(m_1,\ 2m_{12})+L_2(m_{12},\ m_1-m_2)+\lambda_2L_1(\sigma_1,\ 2\sigma_{12})+\lambda_1L_2(\sigma_{12},\ \sigma_1-\sigma_2)$$
$$n_2=L_2(m_2,\ 2m_{12})+L_1(m_{12},\ m_2-m_1)+\lambda_1L_2(\sigma_2,\ 2\sigma_{12})+\lambda_2L_1(\sigma_{12},\ \sigma_2-\sigma_1) \qquad (1.10)$$

The coefficients n_1 and n_2 are in the nature of auxiliary quantities, and they disappear from consideration as soon as relations (1.6) are substituted into (1.5). It is easily seen in this operation that the terms with λ_1 and λ_2, appearing on the right-hand sides of (1.10), will be superfluous; it is therefore possible to use for these quantities the formulas

$$n_1=L_1(m_1,\ 2m_{12})+L_2(m_{12},\ m_1-m_2)$$
$$n_2=L_2(m_2,\ 2m_{12})+L_1(m_{12},\ m_2-m_1) \qquad (1.11)$$

Equations (1.9) and (1.11) will represent the first group of fundamental equations of the theory of shells.

2. Geometric equations. The strain components and the displacement components at an arbitrary point are connected with each other by relations which can be written as follows:

$$e_{\alpha\alpha}=\frac{1}{1+k_1z}\left[\frac{\partial U_\alpha}{A\partial\alpha}+\frac{U_\beta}{AB}\frac{\partial A}{\partial\beta}+k_1U_z\right],\quad e_{\beta\beta}=\frac{1}{1+k_2z}\left[\frac{1}{B}\frac{\partial U_\beta}{\partial\beta}+\frac{U_\alpha}{AB}\frac{\partial B}{\partial\alpha}+k_2U_z\right]$$

$$e_{\alpha\beta}=\frac{1}{1+k_1z}\left[\frac{1}{A}\frac{\partial U_\beta}{\partial\alpha}-\frac{U_\alpha}{AB}\frac{\partial A}{\partial\beta}\right]+\frac{1}{1+k_2z}\left[\frac{1}{B}\frac{\partial U_\alpha}{\partial\beta}-\frac{U_\beta}{AB}\frac{\partial B}{\partial\alpha}\right]\qquad(2.1)$$

$$e_{\alpha z}=\frac{1}{1+k_1z}\left[(1+k_1z)\frac{\partial U_\alpha}{\partial z}-k_1U_\alpha+\frac{1}{A}\frac{\partial U_z}{\partial\alpha}\right]$$

$$e_{\beta z}=\frac{1}{1+k_2z}\left[(1+k_2z)\frac{\partial U_\beta}{\partial z}-k_2U_\beta+\frac{1}{B}\frac{\partial U_z}{\partial\beta}\right],\qquad e_{zz}=\frac{\partial U_z}{\partial z}$$

Using the hypothetic relations of Kirchhoff-Love, namely $e_{zz}=ee_{\alpha z}=e_{\beta z}=0$, we find from the last three equations of the system (2.1)

$$U_\alpha=u+zf_1,\qquad U\ =v+zf_2,\qquad U_z=w\qquad(2.2)$$

where u, v, w are the displacement components of a point of the middle surface, while

$$f_1=k_1u-\frac{1}{A}\frac{\partial w}{\partial\alpha},\qquad f_2=k_2v-\frac{1}{B}\frac{\partial w}{\partial\beta}\qquad(2.3)$$

Substituting (2.2) into the first three equations of the system (2.1), we write them in the form

$$e_{\alpha\alpha}=\varepsilon_1+z\varkappa_1,\qquad e_{\beta\beta}=\varepsilon_2+z\varkappa_2,\qquad e_{\alpha\beta}=\omega+2z\tau\qquad(2.4)$$

where

$$\begin{aligned}\varepsilon_1&=L_1(u,\ v)+k_1w,\qquad \varkappa_1=L_1(f_1,\ f_2)-k_1\varepsilon_1\\ \varepsilon_2&=L_2(v,\ u)+k_2w,\qquad \varkappa_2=L_2(f_2,\ f_1)-k_2\varepsilon_2\\ \omega&=L_2(u-v)+L_1(v,\ -u)\\ 2\tau&=L_1(f_2,\ -f_1)+L_2(f_1,\ -f_2)-k_1L_1(v,\ -u)-k_2L_2(u,\ -v)\end{aligned}\qquad(2.5)$$

represent the deformation components of the middle surface. Relations (2.5) are the second group of the fundamental equations of the shell theory. We note that the expressions for ϵ_1, ϵ_2 and ω are independent of the Kirchhoff-Love hypotheses; they follow immediately from (2.1) and are exact.

The third group of fundamental equations will be furnished by the relations of compatibility of the deformation components of the middle surface as given by Gol'denveizer. If, instead of the reduced components of bending and twisting deformation used by Gol'denveizer, we use the actual components (2.5), the compatibility equations assume the form

$$L_1(\varkappa_2, -2\tau) - L_2(\tau, \varkappa_1 - \varkappa_2) - (k_1 + k_2) L_1(\varepsilon_2, -\omega) + 2k_1 L_2(\tfrac{1}{2}\omega, \varepsilon_1 - \varepsilon_2) = 0$$
$$L_2(\varkappa_1, -2\tau) - L_1(\tau, \varkappa_2 - \varkappa_1) - (k_1 + k_2) L_2(\varepsilon_1, -\omega) + 2k_2 L_1(\tfrac{1}{2}\omega, \varepsilon_2 - \varepsilon_1) = 0$$
$$L_1(\gamma_1, \gamma_2) + L_2(\gamma_2, \gamma_1) + k_2\varkappa_1 + k_1\varkappa_2 = 0 \qquad (2.6)$$

where (2.7)

$$\gamma_1 = L_1(\varepsilon_2, -\omega) - L_2(\tfrac{1}{2}\omega, \varepsilon_1 - \varepsilon_2), \qquad \gamma_2 = L_2(\varepsilon_1, -\omega) - L_1(\tfrac{1}{2}\omega, \varepsilon_2 - \varepsilon_1)$$

Equations (2.6) differ from the corresponding equations of Gol'denveizer by appearance, in the first two equations, of some supplementary terms containing the curvatures k_1 and k_2. The third equation remains unchanged, since the correction involved would be of the order of magnitude of λ^2 in comparison with unity.

3. Equations of elasticity. Disregarding the component σ_z as negligible in comparison with the fundamental stress components σ_α and σ_β, we obtain from Hooke's laws

$$e_{\alpha\alpha} = \frac{1}{E}(\sigma_\alpha - \mu\sigma_\beta), \qquad e_{\beta\beta} = \frac{1}{E}(\sigma_\beta - \mu\sigma_\alpha), \qquad e_{\alpha\beta} = \frac{2(1+\mu)}{E}\sigma_{\alpha\beta} \qquad (3.1)$$

Substituting (1.1) into (3.1) and representing the deformations in the form (2.4), we obtain the elasticity relations of the shell theory in the following simplest form:

$$\begin{aligned} \sigma_1 &= \frac{E}{1-\mu^2}(\varepsilon_1 + \mu\varepsilon_2), & \sigma_2 &= \frac{E}{1-\mu^2}(\varepsilon_2 + \mu\varepsilon_1), & \sigma_{12} &= \frac{E}{2(1+\mu)}\omega \\ m_1 &= \frac{Eh}{1-\mu^2}(\varkappa_1 + \mu\varkappa_2), & m_2 &= \frac{Eh}{1-\mu^2}(\varkappa_2 + \mu\varkappa_1), & m_{12} &= \frac{Eh}{1+\mu}\tau \end{aligned} \qquad (3.2)$$

Formulas (3.2) represent the fourth group of the fundamental equations of the general theory of shells.

Equations (1.9), (2.5), (2.6), (3.2) give us a complete system of fundamental equations of the general theory of shells; for certain boundary conditions along the boundary line of a shell, the system just indicated permits actual problems to be solved.

4. The use of auxiliary functions. Lur'e and Gol'denveizer have shown that if the shell is subjected to loads along the boundary only, i.e. if $q_1 = q_2 = q_3 = 0$, then the equilibrium equations can be satisfied by expressing all stress resultants and couples in terms of four auxiliary functions. Analogous considerations, which reduce to juxtaposition of Equations (1.9) and (2.6) show in the present study that it is superfluous to introduce a fourth function; the relations of the authors mentioned can be represented by means of three functions in the following manner:

$$\begin{gathered}\sigma_1 = h\varkappa_2(a, b, c), \qquad \sigma_2 = h\varkappa_1(a, b, c), \qquad \sigma_{12} = -h\tau(a, b, c) \\ m_1 = -3\varepsilon_2(a, b, c), \qquad m_2 = -3\varepsilon_1(a, b, c), \qquad m_{12} = \tfrac{3}{2}\,\omega(a, b, c)\end{gathered} \tag{4.1}$$

The quantities a, b, c are here auxiliary functions of the coordinates α and β, to be substituted into (2.5) to (2.3) for u, v, w. All stress resultants and couples can be determined from (4.1) on the basis of (1.4).

5. Compatibility equations in terms of stresses. Substituting the deformation components from (3.2) into (2.6), we obtain the compatibility equations for the components of deformation in terms of those of stress

$$L_1(m_1, 2m_{12}) + L_2(m_{12}, m_1 - m_2) - \frac{1}{1+\mu} L_1(m_1 + m_2, 0) + \frac{\lambda_1 + \lambda_2}{1+\mu} \times$$
$$\times L_1(\sigma_1 + \sigma_2, 0) - (\lambda_1 + \lambda_2) L_1(\sigma_1, 2\sigma_{12}) - 2\lambda_1 L_2(\sigma_{12}, \sigma_1 - \sigma_2) = 0 \tag{5.1}$$
$$L_2(m_2, 2m_{12}) + L_1(m_{12}, m_2 - m_1) - \frac{1}{1+\mu} L_2(m_1 + m_2, 0) +$$
$$+ \frac{\lambda_1 + \lambda_2}{1+\mu} L_2(\sigma_1 + \sigma_2, 0) - (\lambda_1 + \lambda_2) L_2(\sigma_2, 2\sigma_{12}) - 2\lambda_2 L_1(\sigma_{12}, \sigma_2 - \sigma_1) = 0$$
$$L_1(p_1, p_2) + L_2(p_2, p_1) - \frac{m_1 - \mu m_2}{R_2 h(1+\mu)} - \frac{m_1 - \mu m_2}{R_1 h(1+\mu)} = 0$$

In the last of these equations

$$\begin{gathered}p_1 = L_1(\sigma_1, 2\sigma_{12}) + L_2(\sigma_{12}, \sigma_1 - \sigma_2) - \frac{1}{1+\mu} L_1(\sigma_1 + \sigma_2, 0) \\ p_2 = L_2(\sigma_2, 2\sigma_{12}) + L_1(\sigma_{12}, \sigma_2 - \sigma_1) - \frac{1}{1+\mu} L_2(\sigma_1 + \sigma_2, 0)\end{gathered} \tag{5.2}$$

Equations (1.9) and (5.1) are a system of six equations for six unknowns; they represent the basic equations for the solution of problems in terms of stresses.

6. Complex equations of V.V. Novozhilov for the case $\mu = 0$. If $\mu = 0$, we substitute

$$t_1 = \sigma_1 - \frac{i}{\sqrt{3}} m_2, \qquad t_2 = \sigma_2 - \frac{i}{\sqrt{3}} m_1, \qquad t_{12} = \sigma_{12} + \frac{i}{\sqrt{3}} m_{12} \tag{6.1}$$

into the equations of equilibrium (1.9) and the equations of compatibility (5.1), which transforms the double system into an equivalent system of three complex equations in terms of the complex stresses t_1, t_2, t_{12}, namely

$$L_1(t_1,\ 2t_{12}) + \left(1 - i\frac{2\lambda_1}{\sqrt{3}}\right)L_2(t_{12},\ t_1 - t_2) + i\frac{\lambda_1+\lambda_2}{\sqrt{3}}L_1(t_2;\ -2t_{12}) + \frac{q_1}{2h} = 0 \tag{6.2}$$

$$L_2(t_2,\ 2t_{12}) + \left(1 - i\frac{2\lambda_2}{\sqrt{3}}\right)L_1(t_{12},\ t_2 - t_1) + i\frac{\lambda_1+\lambda_2}{\sqrt{3}}L_2(t_1,\ -2t_{12}) + \frac{q_2}{2h} = 0$$

$$\frac{i}{\sqrt{3}}[L_1(r_1,\ r_2) + L_2(r_2,\ r_1)] - \frac{t_1}{R_1 h} - \frac{t_2}{R_2 h} + \frac{q_3}{2h^2} = 0$$

where

$$r_1 = L_1(t_2,\ -2t_{12}) - L_2(t_{12},\ t_1 - t_2),\quad r_2 = L_2(t_1,\ -2t_{12}) - L_1(t_{12},\ t_2 - t_1) \tag{6.3}$$

If $\mu \neq 0$, a system of substitutions, analogous to (6.1) which reduces Equations (1.9) and (5.1) to a system of three complex equations, cannot be obtained. Such a possibility could be created only by simplifying these equations at the cost of omitting some small terms of the order of magnitude of λ, leaving in them at the same time some small terms of the same order of magnitude; this would be, however, an inconsistent procedure.

7. Fundamental equations of the theory of plates. Substituting $R_1 = \infty$, $R_2 = \infty$ into (1.9) and (5.1), and taking into account (1.12) and (5.2), we obtain the fundamental equations of the theory of plates in curvilinear coordinates:

$$\begin{aligned} &L_1(\sigma_1,\ 2\sigma_{12}) + L_2(\sigma_{12},\ \sigma_1 - \sigma_2) + \frac{q_1}{2h} = 0 \\ &L_2(\sigma_2,\ 2\sigma_{12}) + L_1(\sigma_{12},\ \sigma_2 - \sigma_1) + \frac{q_2}{2h} = 0 \end{aligned} \tag{7.1}$$

$$\nabla^2(\sigma_1 + \sigma_2) + \frac{1+\mu}{2h}[L_1(q_1,\ q_2) + L_2(q_2,\ q_1)] = 0 \tag{7.2}$$

$$\nabla^2(m_1 + m_2) + \frac{3(1+\mu)}{2h^2}q_3 = 0 \tag{7.3}$$

$$\begin{aligned} &L_1(m_1,\ 2m_{12}) + L_2(m_{12},\ m_1 - m_2) - \frac{1}{1+\mu}L_1(m_1 + m_2,\ 0) = 0 \\ &L_2(m_2,\ 2m_{12}) + L_1(m_{12},\ m_2 - m_1) - \frac{1}{1+\mu}L_2(m_1 + m_2,\ 0) = 0 \end{aligned} \tag{7.4}$$

Equations (7.1) and (7.2) describe the plane state of stress in a plate; they represent two equations of equilibrium and one equation of compatibility of deformation components; Equations (7.3) and (7.4) describe bending of the plate; they consist of one equation of equilibrium and two equations of compatibility.

In the case $q_1 = q_2 = 0$ we introduce an Airy stress function $\Phi(\alpha, \beta)$; with the notations

$$\varphi_1 = L_1(\Phi, 0), \qquad \varphi_2 = L_2(\Phi, 0) \tag{7.5}$$

the solution of Equations (7.1) can be taken in the form

$$\sigma_1 = L_2(\varphi_2, \varphi_1), \quad \sigma_2 = L_1(\varphi_1, \varphi_2), \quad \sigma_{12} = -L_1(\varphi_2, -\varphi_1) = -L_2(\varphi_1, -\varphi_2) \tag{7.6}$$

while Equation (7.2) is reduced in this procedure to the biharmonic equation of the plane problem

$$\nabla^2\nabla^2\Phi = 0 \tag{7.7}$$

Analogously, introducing another function Ψ and using the notations

$$\psi_1 = L_1(\Psi, 0), \qquad \psi_2 = L_2(\Psi, 0) \tag{7.8}$$

we can satisfy Equations (7.4) by setting

$$m_1 = L_1(\psi_1, \psi_2) + \mu L_2(\psi_2, \psi_1), \quad m_2 = L_2(\psi_2, \psi_1) + \mu L_1(\psi_1, \psi_2) \tag{7.9}$$
$$m_{12} = (1-\mu)L_1(\psi_2, -\psi_1) = (1-\mu)L_2(\psi_1, -\psi_2)$$

while (7.3) assumes the form

$$\nabla^2\nabla^2\Psi = -\frac{3}{2h^2}q_3 \tag{7.10}$$

Combining (7.9) with the elasticity relations (3.2) and Expressions (2.3) for the bending and twisting deformations, we find that

$$\Psi = -\frac{Eh}{1-\mu^2}w \tag{7.11}$$

Therefore, instead of (7.10) we will have the equation of Sophie Germain

$$\nabla^2\nabla^2 w = \frac{q}{D} \tag{7.12}$$

The left-hand sides of Equations (7.1) to (7.4) are the principal terms in Equations (1.9) and (5.1) of the general theory of shells. This, as well as the further transformation of these equations in accordance with (7.5) to (7.12), is taken into consideration in the derivation of approximate theories of analysis of shells.

8. Approximations of the theory for the case of shallow shells. Let us consider the state of stress of a shell under the assumption that the stresses due to extension and those due to bending are of the same order of magnitude, so that $\sigma_i \approx m_i$. In this case, we obtain, by omitting in all equations (1.9) and (5.1) small quantities of the order of magnitude of λ as compared with unity, the following approximate equations:

$$\begin{aligned} &L_1(\sigma_1, 2\sigma_{12}) + L_2(\sigma_{12}, \sigma_1 - \sigma_2) + \frac{q_1}{2h} = 0 \\ &L_2(\sigma_2, 2\sigma_{12}) + L_1(\sigma_{12}, \sigma_2 - \sigma_1) + \frac{q_2}{2h} = 0 \end{aligned} \tag{8.1}$$

$$\nabla^2(m_1 + m_2) - \frac{3(1+\mu)}{h}\left(\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2}\right) + \frac{3(1+\mu)}{2h^2} q_3 = 0 \tag{8.2}$$

$$\begin{aligned} &L_1(m_1, 2m_{12}) + L_2(m_{12}, m_1 - m_2) - \frac{1}{1+\mu} L_1(m_1 + m_2, 0) = 0 \\ &L_2(m_2, 2m_{12}) + L_1(m_{12}, m_2 - m_1) - \frac{1}{1+\mu} L_2(m_1 + m_2, 0) = 0 \end{aligned} \tag{8.3}$$

$$\nabla^2(\sigma_1 + \sigma_2) + \frac{m_1 - \mu m_2}{R_2 h} + \frac{m_2 - \mu m_1}{R_1 h} + \frac{1+\mu}{2h}[L_1(q_1, q_2) + L_2(q_2, q_1)] = 0 \tag{8.4}$$

It is not difficult to see that (8.1) to (8.4) correspond to the equations adopted by Vlasov in the theory of shallow shells. We know that these equations have been obtained by Vlasov on the basis of some geometric and static assumptions. The present discussion shows that the theory developed by Vlasov is a mathematically consistent theory accurate to $1 + \lambda \approx 1$ in such cases, when the stresses due to extension and to bending are of the same order of magnitude. The passage from the equations (8.1) to (8.4) to those of Vlasov is materialized by formal utilization of the results of the preceding section for the case under consideration. We find that Equations (8.1) and (8.3) coincide as regards their structure with Equations (7.1) and (7.4); this does not mean, however, that they are identical. The fact is that in the case of $q_1 = 0$, $q_2 = 0$ substitution of (7.6) and (7.9) satisfies Equations (7.1) and (7.4); the former are thus the integrals of the latter. This does not apply to Equations (8.1) and (8.3), since the substitution of (7.6) and (7.9) produces, on the basis of Gauss' condition, on the right-hand sides of (8.1) and (8.3) small quantities differing from zero; e.g. for (8.1) we find

$$\begin{aligned} &L_1(\sigma_1, 2\sigma_{12}) + L_2(\sigma_{12}, \sigma_1 - \sigma_2) = -\frac{1}{R_1}\frac{1}{R_2} L_1(\Phi, 0) \\ &L_2(\sigma_2, 2\sigma_{12}) + L_1(\sigma_{12}, \sigma_2 - \sigma_1) = -\frac{1}{R_1}\frac{1}{R_2} L_2(\Phi, 0) \end{aligned} \tag{8.5}$$

The right-hand sides of (8.5) (as well as the equations obtained from

(8.3)) are small in comparison with the terms which cancel each other on the left-hand side; therefore (7.6) and (7.9) may be regarded with a sufficient degree of accuracy to be the integrals also of Equations (8.1) and (8.3).

Substituting (7.6) and (7.9) into Equations (8.2) and (8.4), we obtain

$$\nabla^4\Psi - \frac{3}{h}\left[\frac{1}{R_1} L_2(\varphi_2, \varphi_1) + \frac{1}{R_2} L_1(\varphi_1, \varphi_2)\right] - \frac{3}{2h^2} q_3 = 0$$
$$\nabla^4\Phi - \frac{1-\mu^2}{h}\left[\frac{1}{R_1} L_2(\psi_2, \psi_1) + \frac{1}{R^2} L_1(\psi_1, \psi_2)\right] = 0 \qquad (8.6)$$

These equations coincide with those of Vlasov if the function Ψ is replaced there by the deflection w according to Formula (7.11). This replacement is equivalent to omitting, in Expressions (2.5) for the components of bending and twisting deformation, the tangential displacements u, v, while retaining the radial displacement w, which is in agreement with the adopted degree of accuracy.

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